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# THE CONCEPT OF NONLINEAR NORMAL MODES AND THEIR APPLICATION IN STRUCTURAL DYNAMICS

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**Abstract:** Due to wide-ranging and successful aplication of linear modal analysis (LMA), not only in the structural dynamics, but practically in all areas of natural and engineering sciences, its simple concept strives to extends to nonlinear dynamical systems. This has led to the formulation of nonlinear modal analysis (NMA), which has a potential to become powerful tool in the analysis of real dynamic systems. This paper presents a comparison of simple linear and nonlinear systems, and the concept of nonlinear normal modes (NNM). For realistic estimation of dynamical parameters experimental modal analysis is neccessary, which can then, in combination with nonlinear analysis, give real results.

**Keywords:** linear modal analysis, nonlinear modal analysis, nonlinear normal modes, experimental modal analysis

# KONCEPT NELINEARNIH NORMALNIH MODOVA I NJIHOVA PRIMJENA U DINAMICI KONSTRUKCIJA

**Sažetak:** Zbog široke i uspješne primjene linearne modalne analize (LMA), ne samo u dinamici konstrukcija, već praktično u svim područjima prirodnih i tehničkih znanosti, nastoji se njezin jednostavan koncept protegnuti na nelinearne dinamičke sustave. To je dovelo do formulacije nelinearne normalne analize (NMA), koja ima potencijal da postane snažan alat u analizi realnih dinamičkih sustava. U ovom radu je prikazana usporedba jednostavnih linearnih i nelinearnih sustava, te koncept nelinearnih normalnih modova (NNM). Za realnu procjenu dinamičkih parametara nužna je eksperimentalna modalna analiza, koja zatim u kombinaciji s nelinearnom analizom može dati realne rezultate.

**Ključne riječi:** linearna modalna analiza, nelinearna modalna analiza, nelinearni normalni modovi, eksperimentalna modalna analiza

## 1. Introduction

Linear modal analysis (LMA) is a well known and effective method for obtaining responses of linear dynamic systems with multiple degrees of freedom. It is based on linear normal modes (LNM). Each LNM is an intrinsic structural property representing the synchronous vibration of the structure at resonance. The linear normal mode is characterized by three parameters, modal shape (structural deformation), natural frequency and damping. One important mathematical property of LNMs is their mutual orthogonality. The finite element method (FEM), LMA and other linear methods have become standard procedures for solving structural dynamics problems. When the accuracy of the solution is not very important, systems can still be treated as linear, although they are mainly not. However, when the accuracy of the predicted response is of vital importance, or when the nonlinear effects are high, a linear analysis becomes unreliable. Expectations in terms of high accuracy of the response become increasingly higher with great advances in the development of computers.

Most engineering structures exhibit nonlinear behavior, which is often of a local character (supports, connections, change in geometry etc.). Solutions of simpler nonlinear problems can be obtained relatively easily using some of the numerical methods. However, when it comes to systems with many degrees of freedom, such computations can take extremely long and are impractical for real situations. Real problems are usually solved by linear methods, while their nonlinear properties are in most cases covered by a simple approximation. This is partly due to the absence of a single theory that can cover general nonlinearity cases [1]. However, in order to obtain real behavior of a system, it is necessary to use nonlinear methods. The concept of nonlinear normal mode (NNM) provides a solid theoretical basis for interpretation of the dynamics of nonlinear systems. In early 1960s, the papers [2,3] presented a concept of nonlinear normal modes resulting from extension of the concept of linear normal modes, which provides a clear conceptual relation between them. While there are many system nonlinearity sources, the NNM literature mainly deals with localized stiffness nonlinearities and distributed (geometrical) nonlinearities. The first numerical method related to NNM is presented in paper [4]. More advanced approaches appear later [5]. The NNM concept was extended to nonconservative systems in 1990s [6,7]. Using analytical methods, NNMs were obtained for systems with linear and nonlinear damping [8]. Recently, different interpretations of the NNM definition resulted in the formulation of several numerical methods for their determination [9].

The complexity of nonlinear modal analysis (NMA) results from the following facts:

- localized nonlinearity can have a significant effect on the entire structure, while some of its parts remain in the linear domain,
- nonlinear effects are usually contained in only a few modes, while the rest behave linearly,
- there is a lack of standardized parameters that can define the degree of nonlinearity in an objective way,
- there is not a simple and approachable way to present a nonlinear response in the form of a well-defined algebraic function.

Due to the absence of a unified nonlinear theory, nonlinear parameters are usually included in the linear frame (LMA). Such an approach ensures compatibility with LMA, but it is not necessarily the best way. There are serious questions [10] on the validity of extending the linear concept to nonlinear systems. For example, bifurcating nonlinear modes are essentially a nonlinear motion, and cannot be considered as an analytical extension of any linear mode. Furthermore, NNMs can be stable and unstable, unlike LNMs, which are always in a state of neutral equilibrium [11].

## 2. Comparison of linear and nonlinear responses on simple examples

The dynamic response of a linear and nonlinear system with one and two degrees of freedom of movement is observed, as shown in Figure 1. In the case of a nonlinear system with a single degree of freedom (SD), motion is described by the Duffing motion equation

$$m\ddot{u} + c\dot{u} + ku + \beta u^3 = f \sin \Omega t, \tag{1}$$

where nonlinear stiffness is represented by the term $\beta u^3$, where $\beta$ is a nonlinear stiffness coefficient. If $\beta = 0$, Equation (1) becomes linear. In both cases these are forced damped oscillations. The type of forcing is harmonic, with frequency $\Omega$.

In the case of a system with two degrees of freedom, motion is described by the following equations

$$\begin{aligned} m_1\ddot{u}_1 + c_1\dot{u}_1 - c_2(\dot{u}_2 - \dot{u}_1) + k_1 u_1 - k_2(u_2 - u_1) - \beta(u_2 - u_1)^3 &= f_1 \sin \Omega t, \\ m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) + \beta(u_2 - u_1)^3 &= f_2 \sin \Omega t. \end{aligned} \tag{2}$$

As in the case of a SD system, previous equations become linear for $\beta = 0$.

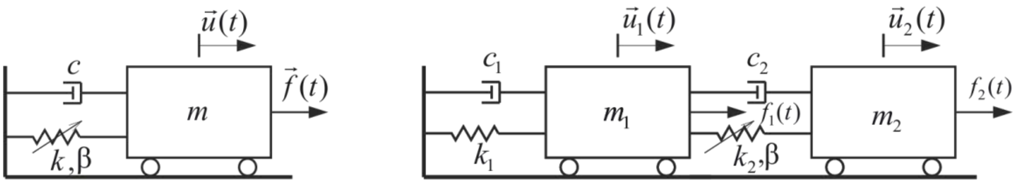


Figure 1. Oscillation systems with single and two degrees of freedom

Stationary solutions of equations (1) and (2) can be obtained e.g. by using the harmonic balance method [11], where the solution is shown by Fourier harmonic series for each degree of freedom.

$$u_i(t) = \sum_{n=1}^{\infty} \left[ a_{in} \sin(n\omega t) + b_{in} \cos(n\omega t) \right]. \tag{3}$$

The fundamental frequency corresponds to the forcing frequency. Truncating the previous series and assuming only the fundamental harmonic, an approximate solution for each degree of freedom can be written in the form

$$u_i(t) = a_i \sin \omega t + b_i \cos \omega t, \tag{4}$$

where $a_i$ and $b_i$ are unknown coefficients.

Since this is an approximate method that does not take into account all terms of the series, there is a residual. Residual orthogonalization (Galerkin weighted residuals) in the interval $(0, 2\pi/\omega)$ gives a system of 2n algebraic equations, whose unknowns are coefficients $a_i$ and $b_i$, which are functions of parameter $\omega$. The solution of this system is a frequency response curve of forced oscillations, and a set of free periodic undamped motions, which represents the modes. This solution can be obtained as a series of points for discrete values of frequency $\omega$. The Newton-Raphson method is most commonly used to solve this system of nonlinear algebraic equations.

Two examples of the analysis of linear and nonlinear oscillations of systems with one and two degrees of freedom are given below.

Figure 2 shows the frequency response of a linear SD system, with parameters

$$m = 1.0,\ k = 4.0,\ c = 0.25,\ f = 1.0,\ \beta = 0,$$

while Figure 3 shows the frequency response of a nonlinear SD system, with parameters

$$m = 1.0,\ k = 4.0,\ c = 0.25,\ f = 1.0,\ \beta = 5,$$

Its natural frequency is $f = 1/T = 0.318 Hz$.

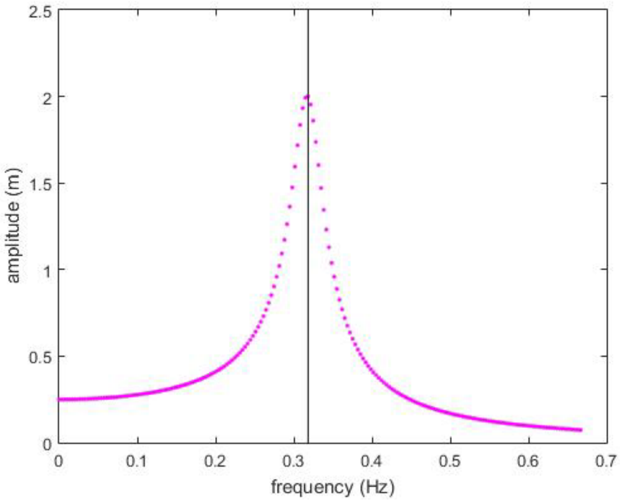


Figure 2. Frequency response of a linear SD system (FRD)

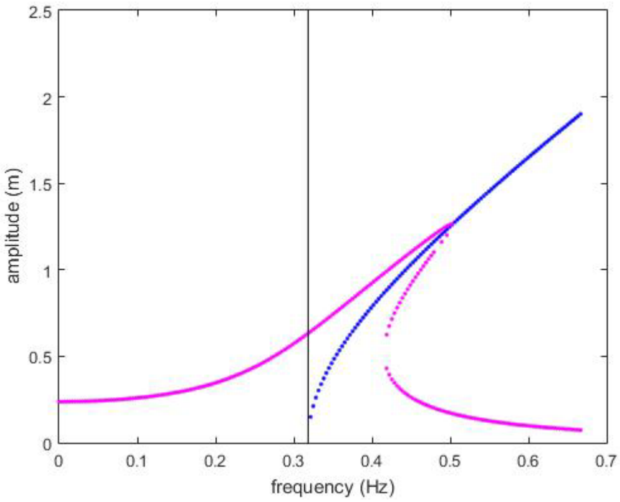


Figure 3. Frequency response of a nonlinear SD system (FRD)

In the case of a linear system, the frequency of free periodic oscillations does not depend on displacement amplitudes or energies, so the modal curves are the vertical lines shown on these diagrams. In linear systems, resonance occurs near natural frequencies, while in nonlinear systems it shifts relative to these frequencies. This shift can be to the right or to the left, depending on the sign of the nonlinear parameter $\beta$. This shift, or bending of the frequency response curve of nonlinear systems indicates that there exists an interval of forcing frequencies between the natural frequency and a certain smaller/greater frequency, where bifurcation points occur. Stability analysis of these stationary solutions, in this frequency interval, would show that the solutions with the largest and the smallest response amplitude are stable, while the intermediate solutions are unstable. For any other forcing frequency outside this interval, the stationary solution is unique and stable [11].

Figures 4 and 5 show the frequency response of a linear system with two degrees of freedom, whose parameters are

$$m_1 = 1.0,\ m_2 = 1.0,\ k_1 = 1.0,\ k_2 = 1.0,\ c_1 = 0.2,\ c_2 = 0.0,\ f_1 = 1.0,\quad f_2 = 0.0,\ \beta = 0.0,$$

while Figures 6 and 7 show the frequency response of a nonlinear system with two degrees of freedom, with parameters

$$m_1 = 1.0,\ m_2 = 1.0,\ k_1 = 1.0,\ k_2 = 1.0,\ c_1 = 0.2,\ c_2 = 0.0,\ f_1 = 1.0,\quad f_2 = 0.0,\ \beta = 0.5.$$

The nonlinear frequency response of the system with two degrees of freedom is analogous to the system with one degree of freedom. It is evident in Figures 4 and 6 that, between the two resonances, there is a frequency at which the amplitude of the first mass is zero. The total energy is therefore contained in the motion of the second mass and damping. This phenomenon is called tuning and has a significant application in mechanical vibration control. In nonlinear systems, energy can be transferred from one mode to another, leading to their change (localization) [12]. An important property of nonlinear systems is that the response can contain harmonics and subharmonics, which means that the response can contain frequencies different from the excitation frequencies. This does not hold true for linear systems, where the response and excitation frequencies are equal.

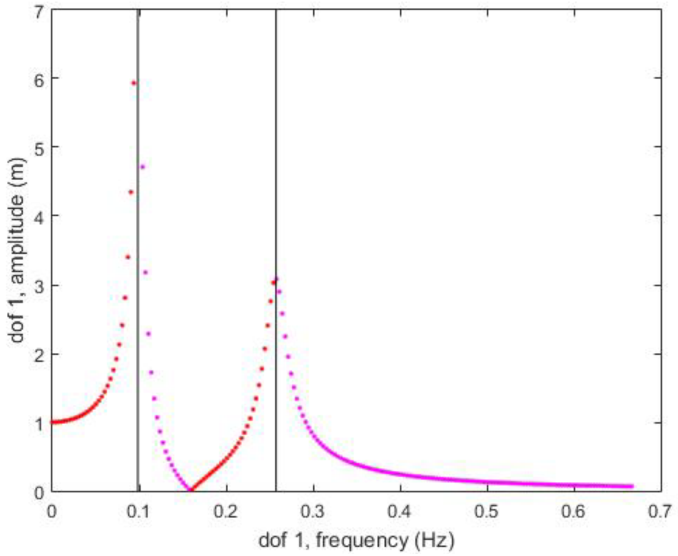


Figure 4. Frequency response of a linear system with two degrees of freedom (stage one)

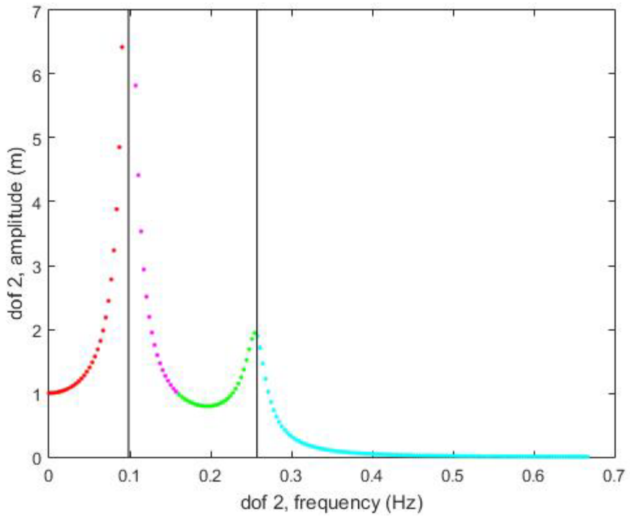


Figure 5. Frequency response of a linear system with two degrees of freedom (stage two)

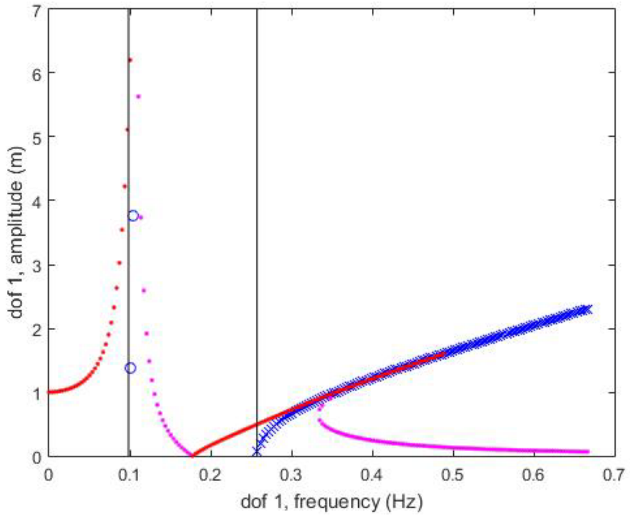


Figure 6. Frequency response of a nonlinear system with two degrees of freedom (stage one)

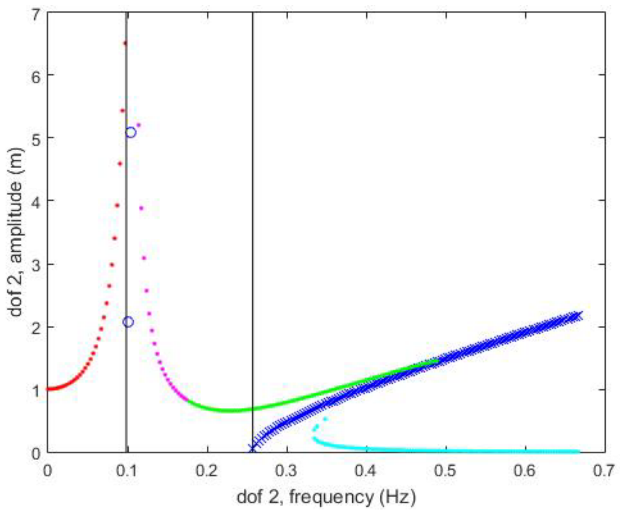


Figure 7. Frequency response of a nonlinear system with two degrees of freedom (stage two)

## 3. The concept of nonlinear normal modes

In the linear theory, proper forms (modes) of discrete or continuous systems are used to uncouple systems of motion equations so that each of them can be separately solved in a space of modal coordinates. Free or forced oscillations of a linear system with multiple degrees of freedom can be then obtained by superposition of modal responses. The number of normal modes (NM) of a linear system cannot be greater than the number of degrees of freedom, while this does not hold true for nonlinear systems due to the occurrence of normal mode bifurcations.

In the case of linear systems, normal modes are independent and there is no energy exchange between them, while in nonlinear systems energy is transferred from one mode to another.

The first definition of NNMs is given by Rosenberg [2]. He defined NNMs as synchronous periodic oscillations, where all degrees of freedom oscillate at the same frequency and the same displacement ratio. This means that all degrees of freedom of the system must reach extreme values at the same time. This definition has two disadvantages. The first is that it cannot apply to nonconservative systems. The second disadvantage lies in the fact that this definition does not cover the possibility of occurrence of internal resonance. This disadvantage can be amended if the definition is extended to non-necessarily synchronous periodic motion, because in the case of internal resonance the motion is still periodic.

The second definition of NNMs, with the extension to damped systems, is given by Shaw and Pierre [6]. They defined NNMs as 2D invariant manifolds (surfaces) in the phase space, which are tangent to the linear planes at the equilibrium position [12].

There are a number of analytical and numerical methods for determining NNMs. Analytical methods worthy of note are the harmonic balance method, a formulation based on energy, the invariant manifold approach, multiple scales method. Numerical methods are more useful than analytical ones because analytical methods are limited to simpler systems with few degrees of freedom and low nonlinearity. The most commonly used numerical methods are Runge-Kutta and Newmark methods, as well as the continuation methods.

According to Vakakis, Kerschen and other authors, results of numerical analysis of NNMs are presented graphically (FRP), Figures 9 and 14. They present NNMs as sequences of discrete points representing NNMs. In the following examples, the software NNMcont [13, 14, 15] was used to obtain NNMs. The NNMs have paths in the phase space that can be presented in two ways. The first way is with modal curves (Figures 10, 12, 15 and 17), and the second way is by using a graph showing the development of NNMs for any degree of freedom (Figures 11, 13, 16 and 18).

An example with two degrees of freedom is analyzed here, the motion of which is described by equations

$$
\begin{aligned}
m_1\ddot{u}_1 + k_1 u_1 - k_2\left(u_2 - u_1\right) - \beta\left(u_2 - u_1\right)^3 = 0 \\
m_2\ddot{u}_2 + k_2\left(u_2 - u_1\right) + \beta\left(u_2 - u_1\right)^3 = 0.
\end{aligned}
\tag{5}
$$

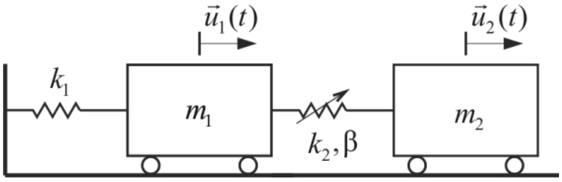


Figure 8. Oscillation system with two degrees of freedom

System parameters are

$$m_1 = 1.0,\ m_2 = 1.0,\ k_1 = 1.0,\ k_2 = 1.0,\ \beta = 0.5.$$

Figure 9 shows the first NNM branch, Figure 10 shows the modal curve for point 1, while Figure 11 shows the time development of displacement and displacement amplitude for the same point. It is visible that lower-energy modes coincide with linear normal modes. The masses oscillate in phase and there is no transfer of energy from one mass to the other, which is evident in Figure 11. Red dots in Figure 9 represent unstable modes, with bifurcation points. In this area, there is more than one mode relative to the linear mode, which is a consequence of internal resonance.

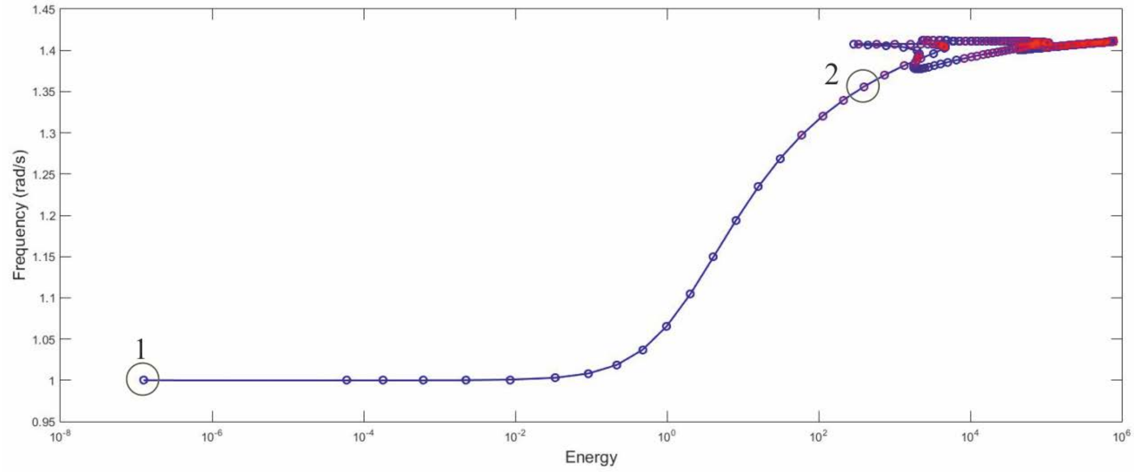


Figure 9. First NNM branch (FRP)

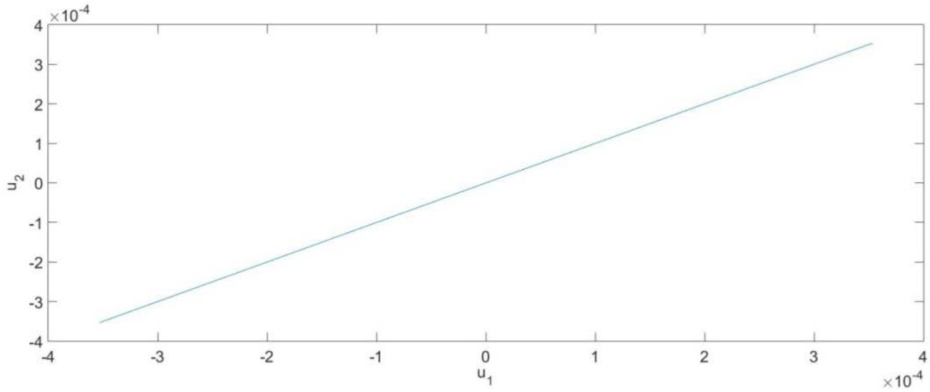


Figure 10. Modal curve for point 1 (first NNM)

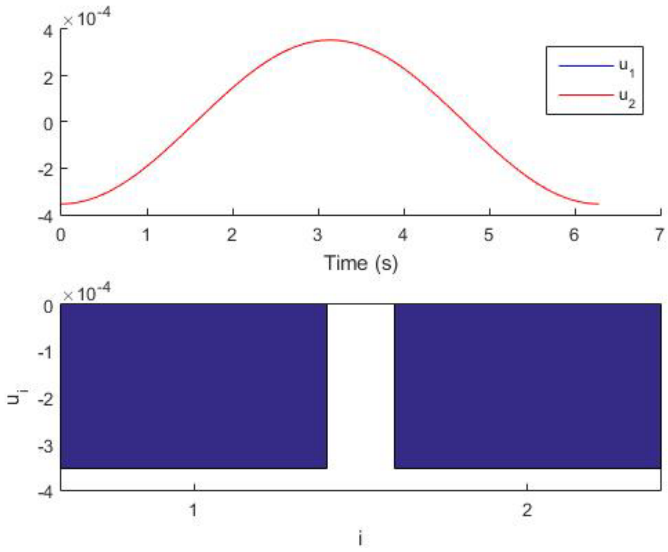


Figure 11. Development of displacement and amplitude in time for point 1

Figure 12 shows the modal curve for point 2 (Figure 9). It is evident that it becomes nonlinear with increasing mode energy. The masses oscillate out of phase, and energy is transferred from the first to the second mass.

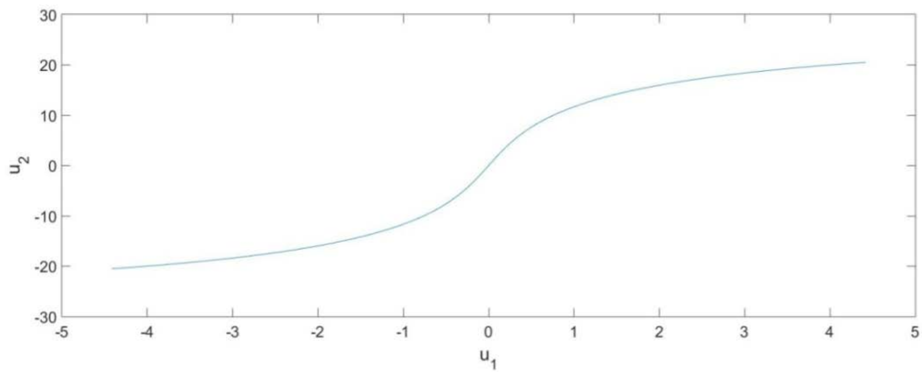


Figure 12. Modal curve for point 2 (first NNM)

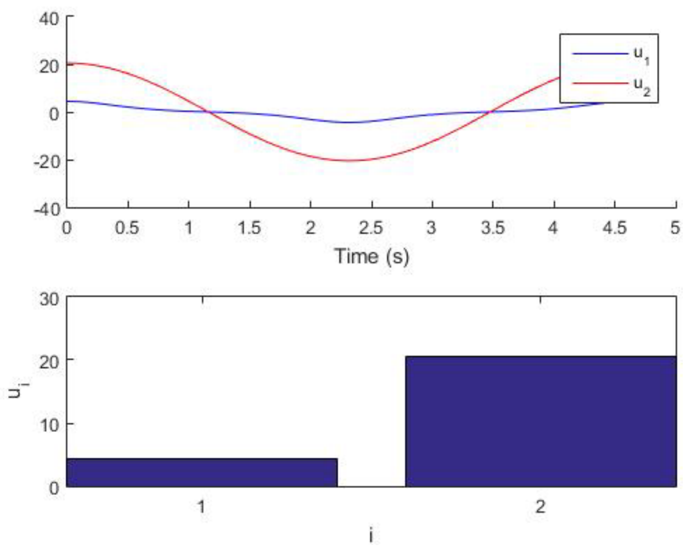


Figure 13. Development of displacement and amplitude in time for point 2

Figure 14 shows the second branch. Since stiffness in this example is adopted using hardening law, the frequency of free periodic oscillations increases with increasing amplitude.

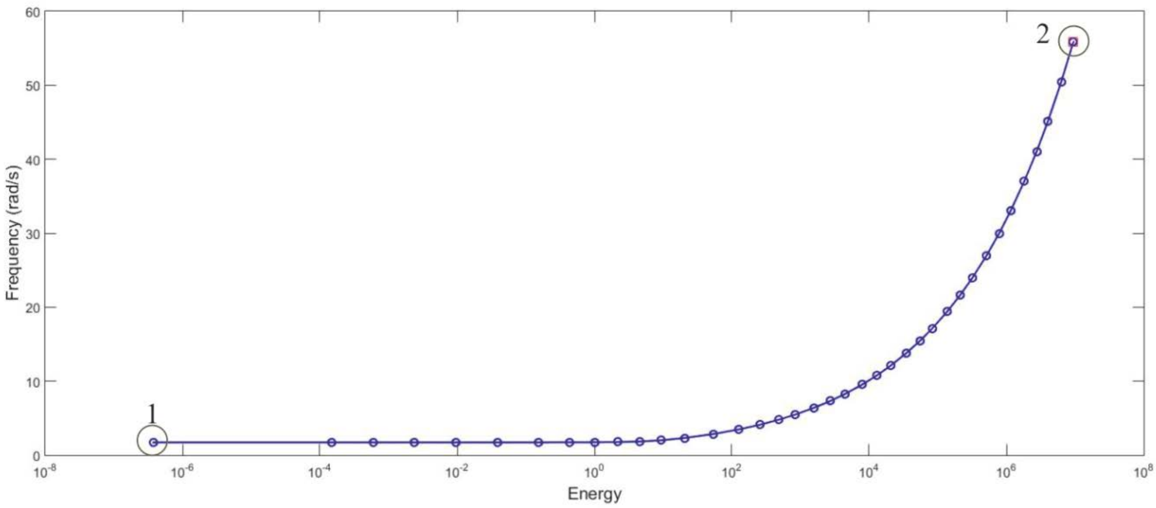


Figure 14. Second NNM branch (FRP)

All modes on this branch are stable and symmetrical. Figure 15 shows the modal curve for point 1, from Figure 14, in the phase space. As with the first branch, it is evident here too that linear and nonlinear modes coincide at low energies. It is visible in Figure 16 that both masses oscillate periodically, out of phase, and that there is a transfer of energy from the first to the second mass.

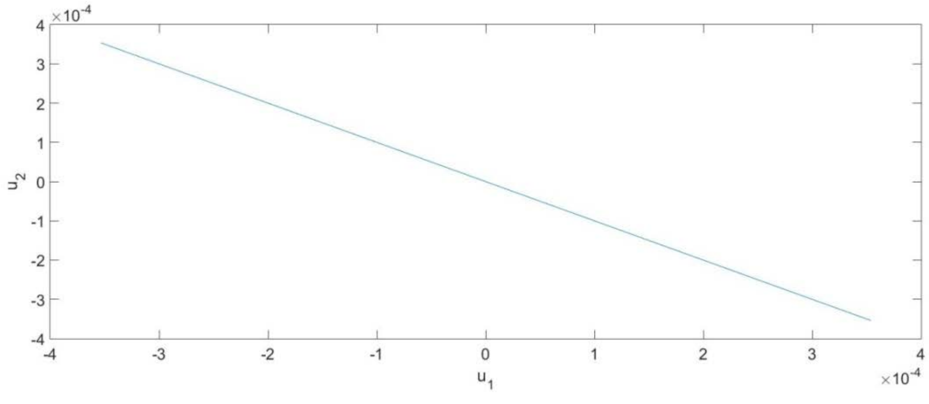


Figure 15. Modal curve for point 1

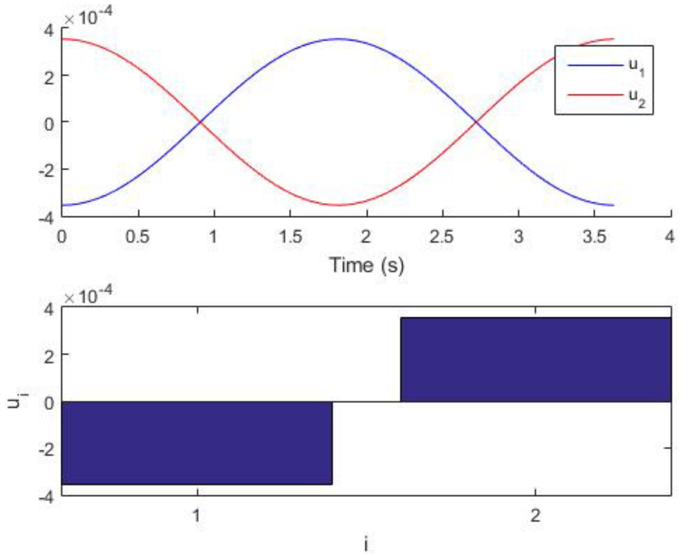


Figure 16. Development of displacement and amplitude in time for point 1

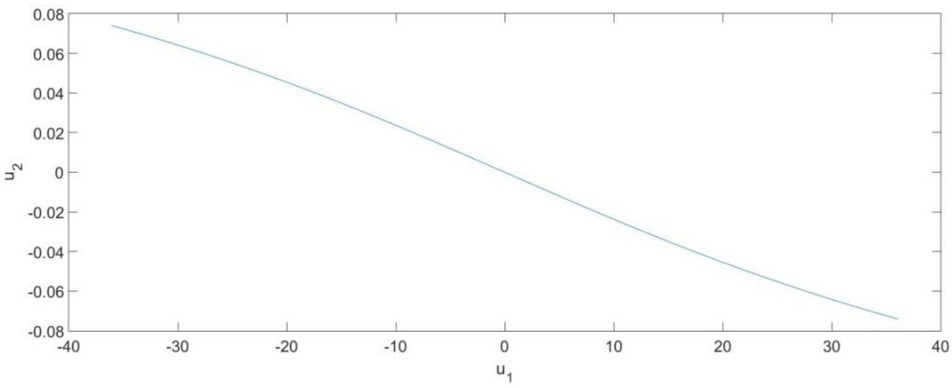


Figure 17. Modal curve for point 2

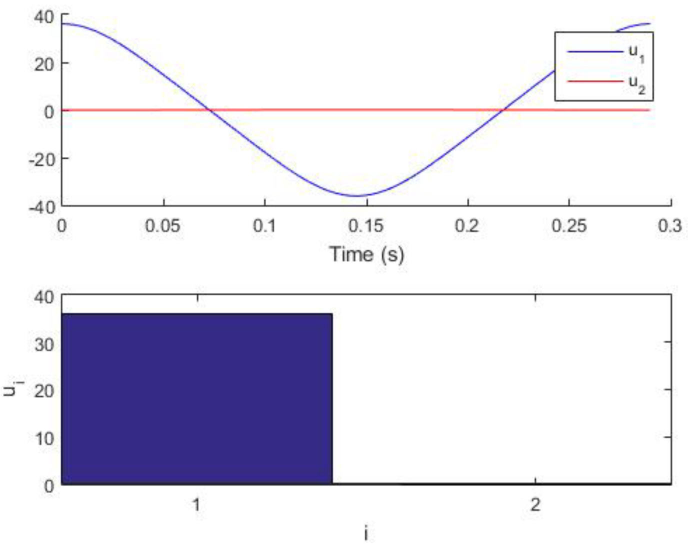


Figure 18. Development of displacement and amplitude in time for point 2

Figure 17 shows the modal curve, in the phase space, for point 2 from Figure 14. In this situation, the total energy of the system is in the first stage, while the amplitude of the second stage equals zero, as can be seen in Figure 18. On the second branch there are no bifurcation points, and thereby no new branches. Oscillations of both masses are out of phase, while their fundamental frequency increases, which leads to a decrease of minimum period.

## 4. Conclusions

In the field of structural dynamics, nonlinear modal analysis (NMA) has emerged as a set of methods that represent the extension of classical linear modal analysis (LMA) for analysis of nonlinear dynamic systems. The development of experimental and analytical modal analysis is keeping pace with each other, and it is desirable to use their combination to obtain dynamic response. Experimental analysis can provide a good estimation of nonlinear parameters (stiffness and damping), which is a basic precondition of any nonlinear analysis, whether it is analytical or numerical. Nonlinear modal analysis represents an interesting field of research that is constantly expanding and bringing new insights and results. It is gradually entering a practical field of application due to its potential in nonlinear modeling of real dynamic systems.

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